Two-dimensional Birkhoff's theorem

Matěj Dostál

Czech Technical University in Prague
Faculty of Electrical Engineering
Department of Mathematics

13/2/2016

Outline

Universal algebra

- Universal algebra is the study of algebraic structures. (Birkhoff, 1935)
- Algebras in "ordinary" universal algebra are sets equipped with operations satisfying some axioms (identities).

Example — monoids

A monoid $(M, *, e)$ is a set M together with a binary operation $*$ and a nullary operation e, satisfying the identities

$$(x * y) * z = x * (y * z),$$
$$e * x = x * e = x.$$

Ordinary Birkhoff's theorem

Commutative monoids

The inclusion CommMon $\hookrightarrow$ Mon is given by the additional identity $x * y = y * x$.

Equational subcategories of $\mathrm{Alg}(\mathcal{T})$

Let $\mathrm{Alg}(\mathcal{T})$ denote algebras for a theory $\mathcal{T}$.

$$\mathcal{A} \hookrightarrow \mathrm{Alg}(\mathcal{T})$$

Every equational subcategory $\mathcal{A}$ of $\mathrm{Alg}(\mathcal{T})$ is closed in $\mathrm{Alg}(\mathcal{T})$ under products, subalgebras, and quotient algebras (HSP).

Birkhoff's HSP theorem

The converse of the above proposition is true; we can detect equational subcategories by their closure properties.

Usage of Birkhoff's theorem

$\mathcal{A}$ = Modular lattices

Lattices satisfying the implication

$$x \leq b \quad \text{implies} \quad x \vee (a \wedge b) = (x \vee a) \wedge b$$

closed under H, S and P: axiomatisable by identities.

$\mathcal{A}$ = Integral domains

Nonzero commutative rings satisfying

$$x * y = 0 \quad \text{implies} \quad x = 0 \quad \text{or} \quad y = 0$$

not closed under HSP: cannot be axiomatised by identities.

Categories with structure

There are important examples of categories equipped with algebraic structure. Basic example: $(\mathrm{Set}, \times, 1)$.

$[2,0]$-categories

A $[2,0]$-category is a category $\mathcal{C}$ together with a functorial binary operation $\otimes$ that is associative up to natural isomorphism

$$\alpha_{ABC} : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C),$$

and a unit I that is an identity for $\otimes$ up to natural isomorphisms.

Categorical universal algebra

- Categorical universal algebra employs category theory in the study of algebra.
- Basic notion on the level of syntax: algebraic theory. (Lawvere, 1963)

Algebraic theory

A theory is a small category $\mathcal{T}$ equipped with some limit structure (e.g. finite products). Corresponds to abstract clones in UA.

Algebra for a theory $\mathcal{T}$

A $\mathcal{T}$-algebra is a functor $\mathcal{T} \to \mathsf{Set}$ that preserves the limit structure of $\mathcal{T}$. Natural transformations correspond to homomorphisms between $\mathcal{T}$-algebras. The category of all $\mathcal{T}$-algebras is denoted by $\mathrm{Alg}(\mathcal{T})$.

Many-sorted algebras

- Slight generalisation: many-sorted algebras. (Birkhoff, 1970)
- Algebras have an underlying set for each specified sort, the operations are sorted.

Example – directed graphs

A directed graph (E, V, s, t) is a set E of the edge sort, a set V of the vertex sort, and two unary operations

$$s : E \to V,$$
$$t : E \to V,$$

representing the source and target maps.

Examples of algebraic theories

Directed graphs — empty limit structure

$$e \underset{t}{\overset{s}{\rightrightarrows}} v \qquad E_A \underset{t_A}{\overset{s_A}{\rightrightarrows}} V_A$$

Contains only unary operations: no need for products.

Monoids — finite product structure

$$g \times g \xrightarrow{*} g \qquad G \times G \xrightarrow{*} G$$

All the derived operations are present in the theory.

Categories with structure

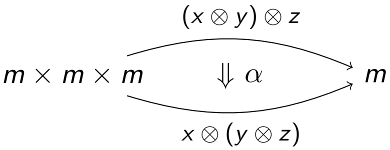
There are important examples of categories equipped with algebraic structure. Recall: $(\mathrm{Set}, \times, 1)$.

$[2,0]$-categories

A $[2,0]$-category is a category $\mathcal{C}$ together with a functorial binary operation $\otimes$ that is associative up to natural isomorphism

$$\alpha_{ABC} : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C),$$

and a unit I that is an identity for $\otimes$ up to natural isomorphisms.



Enriched algebraic theories

Category-enriched categories

Ordinary category $\mathcal{C}$: objects $\mathrm{ob}(\mathcal{C})$, hom-sets $\mathcal{C}(A, B)$.
2-category $\mathcal{C}$: objects $\mathrm{ob}(\mathcal{C})$, hom-categories $\mathcal{C}(A, B)$.

Algebraic theory enriched in categories

A theory is a small 2-category $\mathcal{T}$ equipped with finite products.

Algebra for a theory $\mathcal{T}$

A $\mathcal{T}$-algebra is a 2-functor $\mathcal{T} \to$ Cat that preserves finite products. Natural transformations correspond to homomorphisms between $\mathcal{T}$-algebras. The 2-category of all $\mathcal{T}$-algebras is denoted by $\mathrm{Alg}(\mathcal{T})$.

2-dimensional identities

What are identities?

We do not want to specify identities of the form

$$(A \otimes B) \otimes C = A \otimes (B \otimes C).$$

We are interested in identities between rewrite "strategies", e.g.

$$((A \otimes B) \otimes C) \otimes D \;\overset{\longrightarrow}{\underset{\longrightarrow}{\parallel}}\; A \otimes (B \otimes (C \otimes D))$$

There are two ways to use the "associator" rewrite rule. The identity is one of the axioms of a monoidal category.

2-dimensional HSP theorem

Problem

Characterise equational subcategories

$$\mathcal{A} \hookrightarrow \mathrm{Alg}(\mathcal{T}),$$

where identities glue rewrite strategies together (not terms).

Result (MD, published 2016)

The equational subcategories of $\mathrm{Alg}(\mathcal{T})$ are precisely the subcategories that are closed in $\mathrm{Alg}(\mathcal{T})$ under HSP and "directed unions".

The proof heavily depends on the results of (Bourke, Garner 2014) concerning 2-dimensional factorisation systems and exactness.

References

- MD: A 2-dimensional Birkhoff's theorem, Theory Appl. Categ. (2016)
- Adámek, Rosický, Vitale: Algebraic theories, Cambridge Tracts in Mathematics 184 (2011)
- Birkhoff: On the structure of abstract algebras, 1935
- Bourke, Garner: Two-dimensional regularity and exactness, J. Pure Appl. Algebra 218 (2014), 1346-1371